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INSTABILITY OF LIQUID SURFACES AND THE FORMATION OF DROPS

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I. Introduction

It has long been known that certain flows of liquids with free boundary surfaces are unstable. For steady flows this means that a small perturbation introduced at a point of the flow produces a disturbance which increases in the flow direction until it completely alters the motion. This mechanism was used by Rayleigh to describe the breakup of a steady jet of liquid into drops. In the case of unsteady flows, instability implies that certain initial perturbations will increase in time until they significantly change the flow. Taylor showed that the plane free boundary of a semi-infinite liquid is unstable if the liquid moves with constant acceleration normal to the surface, with the acceleration directed into the liquid. Previously Penney and Pryce had shown that the surface of a pulsating underwater explosion bubble is likewise unstable. In this case, also, the fluid moves with accelerated motion normal to the surface.

With these results in mind, we wish to account for the formation of drops resulting from the breakup of accelerated plane, cylindrical and spherical layers of liquid. These phenomena occur whenever explosives are detonated in or near liquids. For example, the spray dome produced by an underwater explosion results from a layer of water thrown into the air by the shock wave from the explosion. The present theory attempts to describe the breakup of this layer into drops of spray. As a second example, consider the production of an aerosol (i.e. dispersion of liquid drops in air) by a

bomb containing an explosive surrounded by a layer of liquid. The high pressure gases resulting from the explosion propel the liquid layer outward, and it ultimately breaks up into drops. The present theory also attempts to account for this breakup.

In our proposed mechanism of breakup we assume that there is a zero order accelerated flow of the liquid layer. The first order perturbation of this flow, which satisfies linear equations, is represented as a series or integral of normal modes. Some of the modes are found to be unstable in the sense that their amplitudes increase indefinitely with time. Among these, one or more are most unstable, in the sense that they grow most rapidly. In the case of constant acceleration the unstable modes grow exponentially, and the most unstable modes have the largest exponents. The most unstable modes are assumed to be responsible for the breakup of the layer into pieces, which then become spherical due to surface tension. The number of pieces depends upon the number of positive and negative regions into which the most unstable mode function is decomposed by its nodal lines. Thus it is possible to estimate the number and size of the resulting drops.

In the following section (II) the problem is formulated for the case of a plane layer. In section III the zero order solution is considered, and in section IV the first order perturbation is obtained. This perturbation solution

is analyzed in section V, where the formation of drops is discussed. Practical conclusions and recommendations are presented in section VI. Corresponding results for cylindrical and spherical layers will be given in another paper.

II. Formulation

We consider the motion of a plane slab or layer of liquid which separates two gases at different pressures. The liquid will be assumed to be incompressible and inviscid, and the motion of the gases will be neglected. The initial positions and shapes of the bounding surfaces will be prescribed, as well as the initial velocity of the liquid. The problem is to determine the subsequent positions and shapes of these surfaces, and the motion of the liquid.

Assuming that the motion is irrotational, the velocity $u(x,y,z,t)$ of the liquid is derivable from a potential $\phi(x,y,z,t)$ which satisfies Laplace's equation. Thus

$$1. \qquad \vec{u} = -\nabla\phi$$

$$2. \qquad \nabla^2\phi = 0 \quad .$$

The pressure $p(x,y,z,t)$ in the liquid can be expressed in terms of ϕ, the constant density ρ, and the acceleration of gravity g by Bernoulli's equation

$$3. \qquad p\rho^{-1} = \phi_t - gz - \frac{1}{2}(\nabla\phi)^2 \quad .$$

It is assumed that the positive z axis points vertically upward.

The boundary surfaces are assumed to be given by

4. $$z = F_i(x,y,t) \qquad i = 1,2 \;.$$

The subscript $i = 1$ corresponds to the upper surface. The gas pressure on both surfaces is assumed to be independent of x and y, and is denoted by p_i $(i = 1,2)$. Then the dynamic and kinematic boundary conditions may be written respectively as

5. $$\phi_t - gF_i - \frac{1}{2}(\nabla\phi)^2 - (-1)^i\rho^{-1}T_i\nabla^2F_i = \rho^{-1}p_i \quad \text{on } z = F_i \qquad (i = 1,2)$$

6. $$\phi_z - \phi_x F_{ix} - \phi_y F_{iy} + F_{it} = 0 \quad \text{on } z = F_i \qquad (i = 1,2) \;.$$

In (5), T_i is the surface tension of the surface i. The initial conditions will be introduced when they are needed.

If the flow is bounded by a cylindrical tube parallel to the z-axis, the condition $\frac{\partial\phi}{\partial n} = 0$ is imposed on this tube.

If the lower surface $z = F_2$ is a rigid surface with prescribed motion, the above formulation applies provided (5) is omitted for $i = 2$. Therefore this problem will also be considered, since it is so similar to the original problem.

III. Zero Order Solution

If the bounding surfaces are initially planes normal to the z-axis, and if the initial velocity is parallel to the z-axis and independent of x and y, the solution will be independent of x and y. Then, if u(t) denotes the z-component of velocity, we have from (1, 2)

7. $$\phi^{\circ} = -u(t)z + b(t) \quad .$$

Since F_i° depends on t only, we have from (5-7), denoting t differentiation by a dot,

8. $$\dot{b} = (\dot{u}+g)F_i^{\circ} + \frac{1}{2}u^2 + \rho^{-1}p_i \qquad i = 1,2$$

9. $$u = F_{it}^{\circ} \qquad i = 1,2 \quad .$$

From (9) we have, if h denotes the initial separation between the surfaces

10. $$F_2^{\circ} = F_1^{\circ} - h \quad .$$

From (8), eliminating $\dot{b}$, we obtain using (10)

11. $$(\dot{u}+g)\rho h = p_2 - p_1 \quad .$$

Equation (11) yields $\dot{u}$, and therefore u if its initial value is given, provided $p_2 - p_1$ is constant or a known function of t (or even of u and t). If the pressure difference is constant, $\dot{u}$ is constant. However if $p_2 - p_1$ depends upon F_2°, then (11) becomes a second order equation for $F_2^{\circ}(t)$. Finally b is determined from (8), except for an inessential additive constant.

In the problem in which $F_2^{\circ}(t)$ is a prescribed moving rigid surface, u and F_1° are given in terms of F_2° by (9, 10), and $\dot{b}$ by (5) with i = 1, while (5) for i = 2 is omitted, as well as its consequence (11).

IV. First Order Perturbation

Suppose that the initial boundary surfaces differ slightly from planes, and or that the initial velocity differs slightly from the constant z-component assumed in the above solution. Then we may expect the subsequent solution to differ slightly from the above solution, which we will henceforth call the zero order or unperturbed solution. If ε is a measure of the maximum deviations of the initial data, we assume that the solution may be written in the form

12. $$\phi = \phi^{0} + \varepsilon\phi^{1} + \dots , \qquad F_i = F_i^{0} + \varepsilon F_i^{1} + \dots .$$

From (2, 12) we see that ϕ^1 also satisfies Laplace's equation. Now inserting (12) into (5, 6) we obtain for the first order perturbation ϕ^1 and F^1 the equations

13. $$\phi_t - (\dot{u}+g)F_i + u\phi_z - (-1)^i \rho^{-1} T_i \nabla^2 F_i = 0 \quad \text{on } z = F_i^0 \quad (i = 1,2)$$

14. $$\phi_z + \dot{F}_i = 0 \quad \text{on } z = F_i^0 \quad (i = 1,2) .$$

In (13, 14) and the sequel the superscript one on ϕ and F is omitted.

To solve Laplace's equation for ϕ we assume that ϕ is a product of a function of z and t multiplied by a function $\psi(x,y)$. We find

15. $$\phi = [g_1(t)e^{kz} + g_2(t)e^{-kz}]\psi(x,y)$$

16. $$(\nabla^2 + k^2)\psi = 0 \quad .$$

If the flow is unbounded in the x,y plane, then (16) has a bounded solution for every real value of k, and the general solution for ϕ is an integral with respect to k of the expression in the right of (15), in which g_1 and g_2 depend upon k.

However, if the flow is bounded by a rigid cylinder parallel to the z-axis, and if this cylinder intersects the x,y plane in a curve C, then the function ψ must also satisfy the condition

17. $$\frac{\partial \psi}{\partial n} = 0 \quad \text{on } C \quad .$$

If C is closed, then (16, 17) have solutions only for a denumerable set of positive values of k^2, and for $k^2 = 0$. In this case the general solution for ϕ is a series of terms of the form given by (15), summed over the various values of k.

In order to complete the solution we must find g_1, g_2, F_1 and F_2. To this end we assume

18. $$F_i = f_i(t)\psi(x,y) \qquad i = 1,2 \quad .$$

Now, inserting (15, 18) into (13, 14) and making use of (16) we obtain

19. $$\dot{g}_1 e^{kF_i^0} + \dot{g}_2 e^{-kF_i^0} - (\dot{u}+s) f_i + ukg_1 e^{kF_i^0} - ukg_2 e^{-kF_i^0}$$
$$+ (-1)^i k^2 \rho^{-1} T_i f_i = 0 \qquad (i = 1,2)$$

$$20. \qquad k g_1 e^{kF_i^0} - k g_2 e^{-kF_i^0} + \dot{f}_i = 0 \qquad (i = 1,2) \quad .$$

From (20) we find g_1 and g_2 in terms of $\dot{f}_1$ and $\dot{f}_2$:

$$21. \qquad g_1 = -k^{-1} e^{-kF_1^0} (\dot{f}_1 - e^{-kh} \dot{f}_2)(1 - e^{-2kh})^{-1}$$

$$22. \qquad g_2 = k^{-1} e^{kF_1^0} (\dot{f}_1 - e^{kh} \dot{f}_2)(1 - e^{2kh})^{-1} \quad .$$

Now using (21, 22) in (19), and combining the two equation (19), we obtain

$$23. \quad (\ddot{f}_1 - e^{-kh} \ddot{f}_2) + k(\dot{u}+g)(f_1 - e^{-kh} f_2) + k^3 T_1 \rho^{-1} (f_1 + T_1^{-1} T_2 e^{-kh} f_2) = 0$$

$$24. \quad (\ddot{f}_1 - e^{kh} \ddot{f}_2) - k(\dot{u}+g)(f_1 - e^{kh} f_2) - k^3 T_1 \rho^{-1} (f_1 + T_1^{-1} T_2 e^{kh} f_2) = 0 \quad .$$

Equations (23, 24) are a pair of linear homogeneous second order ordinary differential equations for f_1 and f_2. The coefficients are constant only if $\dot{u}$ is constant, which case we shall treat in detail. First however, we wish to point out the limiting forms which (23, 24) attain when h becomes infinite, namely

$$25. \qquad \ddot{f}_1 + k(\dot{u}+g + k^2 T_1 \rho^{-1}) f_1 = 0$$

$$26. \qquad \ddot{f}_2 - k(\dot{u}+g - k^2 T_2 \rho^{-1}) f_2 = 0 \quad .$$

Either of these is the equation for the perturbation of the surface of a semi-infinite region of liquid, in agreement with Taylor's result. We note that for sufficiently large constant $\dot{u}$ one of the above equations will have a growing exponential solution while the other will not.

If $\dot{u}$ is constant we seek solutions of (23, 24) of the form

27. $$f_i = A_i e^{\alpha t} \qquad i = 1,2 \quad .$$

Inserting (27) into (23, 24) and simplifying leads to an equation for α and another for A_2/A_1, namely

28. $$\alpha^4 + \alpha^2 k^3 \rho^{-1}(T_1 + T_2)\coth kh$$
$$- k^2(\dot{u}+g + k^2 T_1 \rho^{-1})(\dot{u}+g - k^2 T_2 \rho^{-1}) = 0$$

29. $$A_2/A_1 = e^{kh}(\alpha^2 + k[\dot{u}+g] + k^3 T_1 \rho^{-1})(\alpha^2 + k[\dot{u}+g] - k^3 T_2 \rho^{-1})^{-1} \quad .$$

The solutions of (28) for α^2 are given by (when $T_1 = T_2$)

30. $$\rho h^3 T_1^{-1} \alpha^2 = -x^3 \coth x \pm x(\beta^2 + x^4 \sinh^{-2} x)^{1/2} \quad .$$

The abbreviations used in (30) are

31. $$x = kh \quad , \qquad \beta = \rho h^2 T_1^{-1} |\dot{u}+g| \quad .$$

Since there are four values of α, each f_i is a sum of four terms of the form in (27).

In the problem in which $F_2(t)$ is the prescribed motion of a plane rigid surface, $F_2^1(t)$ is zero. All equations

through (24) apply, but (13, 18, 19) hold only for $i = 1$ and $f_2 = 0$ in (21, 22). Instead of (23, 24) we obtain for f_1 the equation

$$23'. \qquad \ddot{f}_1 + k \tanh kh \, (\dot{u} + g + k^2 T_1 \rho^{-1}) f_1 = 0 \quad .$$

For infinite h this also has the limiting form (25).

V. Instability and Drop Formation

If α^2 is negative, α is pure imaginary and the corresponding exponential is oscillatory. This occurs when the negative sign is chosen in (30). Only when the plus sign is chosen is there a range of x for which α^2 is positive, and thus one value of α is real and positive leading to a growing exponential. In this range there is a greatest value of α corresponding to a particular value of x which we will call x_{max}, since the exponential corresponding to this value of x grows most rapidly.

To find x_{max} we first note that

$$32. \qquad x^4 \sinh^{-2} x < 1.3 \quad .$$

Thus if $\beta >> \sqrt{1.3} = 1.14$, (30) becomes (using the plus sign)

$$33. \qquad \rho h^3 T_1^{-1} \alpha^2 = x(\beta - x^2 \coth x) \quad .$$

From (33), α^2 is positive for x between zero and $\beta^{1/2}$, approximately. It has its maximum at

34. $$x_{max} = \left(\frac{\beta}{3}\right)^{1/2} = h\left[\frac{\rho|\dot{u}+g|}{3T_1}\right]^{1/2} .$$

Recalling that by (31), $x = kh$, we may introduce k_{max} which is the value of k corresponding to x_{max}, and thus to maximum instability. From (34) we have

35. $$k_{max}^2 = \frac{\rho|\dot{u}+g|}{3T_1} .$$

To find the number of pieces into which the layer breaks up due to the growth of the term corresponding to k_{max}, we must examine the function $\psi(x,y)$ corresponding to k_{max}. It is clear that if the layer is unbounded in the x,y plane there will be many functions $\psi(x,y)$ corresponding to k_{max}. On the other hand, if the region is bounded (17) applies and there may be no solution ψ corresponding exactly to k_{max}, although there will be many functions ψ corresponding to values of k near k_{max}. Thus in both cases there will be many functions ψ with exactly or approximately the same value of k ($\approx k_{max}$) and therefore all the corresponding terms will grow at about the same (maximum) rate. Consequently the exact manner of breakup will depend upon the extent to which these various terms are excited by the initial perturbation.

Nevertheless it is possible to make a rough estimate of the number of pieces into which the layer breaks. First, in the case of an unbounded region, one solution ψ corresponding to k_{max} is

$$36. \qquad \psi = e^{2\pi i(x/\lambda_1 + y/\lambda_2)}$$

where

$$37. \qquad \left(\frac{2\pi}{\lambda_1}\right)^2 + \left(\frac{2\pi}{\lambda_2}\right)^2 = k_{max}^2 \quad .$$

We now consider the regions of the x,y plane in which the real part of ψ is positive (or negative). These are rectangles bounded by nodal lines, and their dimensions are $\frac{\lambda_1}{2}$ and $\frac{\lambda_2}{2}$, with area $\frac{\lambda_1\lambda_2}{4}$. If λ_1 and λ_2 are related by (37), the area of such a rectangle is minimized when $\lambda_1 = \lambda_2 = \frac{2\pi\sqrt{2}}{k_{max}}$. Thus the minimum area of such a rectangle is

$$38. \qquad \frac{\lambda_1^2}{4} = \frac{2\pi^2}{k_{max}^2} = \frac{6\pi^2 T_1}{\rho|\dot{u}+g|} \quad .$$

It is to be expected that (38) will represent roughly the area of a piece into which the layer breaks up since all parts of the surface in this region move in the same direction. The volume of a piece will be given by (38) multiplied by the thickness h. The radius r of the sphere into which the piece will ultimately deform is then (using (11))

$$39. \qquad r = \left[\frac{9\pi T_1 h}{2\rho|\dot{u}+g|}\right]^{1/3} = \left[\frac{9\pi T_1 h^2}{2|p_2-p_1|}\right]^{1/3} \quad .$$

The same results are obtained if the layer is bounded by a tube of rectangular cross-section, and they will also apply for tubes of almost any other shape.

VI. Conclusions and Recommendations

On the basis of (39) three means are available for diminishing drop size, namely

a) Reducing the layer thickness h

b) Reducing the surface tension T_1 by using wetting agents

c) Increasing the explosive pressure p_2.

Of these the first two seem reasonable, while the last appears impractical.

We note from (39) that diminishing h has a twofold effect: the acceleration is greater and the pieces are thinner. It would seem that the reduction in drop size, due to decreasing h, with the consequent reduction in settling out of the drops would more than compensate for the diminution of liquid in the bomb, for a certain range of h. The theory seems to be too crude to allow calculation of an optimum h, but this could be explored experimentally.

The incorporation of wetting agents in the bomb, either mixed with the liquid or in a thin layer separated by a fragile membrane, seems promising. If only one surface can be wet it should be the inner surface.

If the formula (39) is used to calculate the size of drops produced by a bomb, it is necessary to use a value of h less than the original value due to radial expansion of the liquid layer, and its consequent thinning. One might guess that the layer radius increases tenfold, and the thickness h consequently diminishes 100 fold before breakup

occurs. More precise results will be given in the forthcoming paper on the breakup of spherical and cylindrical layers.

To estimate the time required for breakup to occur, we assume that the surface perturbation, given essentially by (27), reaches the value h. This gives for the breakup time

40. $$t = \frac{1}{\alpha_{max}} \log \frac{h}{A} = \left[\frac{27T_1}{4\rho|\dot{u}+g|^3}\right]^{1/4} \log \frac{h}{A}$$

$$= \left[\frac{27T_1\rho^2h^3}{4|p_2-p_1|^3}\right]^{1/4} \log \frac{h}{A} \quad .$$

In (40) A is the initial perturbation of the surface, α_{max} was obtained from (33, 34), and the last equality follows from (11).